

Chapter 5. Wave Propagation

Notes:

- *Most of the material presented in this chapter is taken from Jackson, Chap. 7, and Choudhuri, Chap. 12, 13, and 14.*

5.1 Plane Waves in a Non-conducting Medium

In cases where there are no sources (i.e., $\rho = \mathbf{J} = 0$), Maxwell's equations simplify to

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 0, & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= 0.\end{aligned}\tag{5.1}$$

Since any signal can be expanded with a Fourier series or a Fourier transform (see section 2.1.1), then it is sufficient, in principle, to limit our investigation to solutions exhibiting a time dependence of the form $e^{-i\omega t}$. The global solution, for say the electric field, will be expressed as a superposition of such exponential solutions through a Fourier pair. For example,

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \frac{1}{\sqrt{2\pi}} \int \mathbf{E}(\mathbf{x}, \omega) e^{-i\omega t} d\omega \\ \mathbf{E}(\mathbf{x}, \omega) &= \frac{1}{\sqrt{2\pi}} \int \mathbf{E}(\mathbf{x}, t) e^{i\omega t} dt.\end{aligned}\tag{5.2}$$

For uniform, linear, and isotropic media where $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$, the Faraday and Ampère-Maxwell laws become

$$\begin{aligned}\nabla \times \mathbf{E} - i\omega \mathbf{B} &= 0 \\ \nabla \times \mathbf{B} + i\omega \mu \epsilon \mathbf{E} &= 0,\end{aligned}\tag{5.3}$$

where it is understood that the electromagnetic fields present in these equations are $\mathbf{E}(\mathbf{x}, \omega)$ and $\mathbf{B}(\mathbf{x}, \omega)$. Upon taking the curl of the first, and inserting the second, of equations (5.3) we get

$$\nabla \times (\nabla \times \mathbf{E}) - i\omega \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} - \omega^2 \mu \epsilon \mathbf{E} = 0,\tag{5.4}$$

but since $\nabla \cdot \mathbf{E} = 0$, we have

$$(\nabla^2 + \omega^2 \mu \epsilon) \mathbf{E}(\mathbf{x}, \omega) = 0.\tag{5.5}$$

A similar exercise on the second of equations (5.3) yields a corresponding relation for the magnetic induction

$$(\nabla^2 + \omega^2 \mu \epsilon) \mathbf{B}(\mathbf{x}, \omega) = 0. \quad (5.6)$$

(We could have obtained equations (5.5) and (5.6) from our more general derivations leading to equations (4.53) and (4.54).) A solution to these equations is the so-called **plane wave** with a functional form (using the electric field as an example)

$$\mathbf{E}(\mathbf{x}, \omega) = \mathcal{E}_1 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}, \quad (5.7)$$

with \mathbf{k} the wave vector. Assuming, for simplicity, that $\mathbf{k} = k \mathbf{e}_x$, then we find upon insertion of equation (5.7) into equation (5.5) that

$$k = \omega \sqrt{\mu \epsilon}. \quad (5.8)$$

Such a linear relationship between the angular frequency and the wave number is characteristic of non-dispersive media. The **phase velocity** of the plane wave is

$$v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu \epsilon}} = \frac{c}{n}, \quad (5.9)$$

with the **index of refraction** defined by

$$n = \sqrt{\frac{\mu}{\mu_0} \frac{\epsilon}{\epsilon_0}}, \quad (5.10)$$

which is independent of frequency for non-dispersive media. Substituting equation (5.9) for the angular frequency, a more general solution to the wave equation for a given wave number k (with $\mathbf{k} = \pm k \mathbf{e}_x$) can be written as

$$\mathbf{E}_k(x, t) = \mathcal{E}_1(k) e^{ik(x - v_p t)} + \mathcal{E}_2(k) e^{-ik(x + v_p t)}. \quad (5.11)$$

A superposition of such solutions for different wave number values (or frequencies) with the Fourier transform will yield the general, and expected, solution for the wave equation (for non-dispersive media). That is,

$$\mathbf{E}(x, t) = \mathbf{E}_1(x - v_p t) + \mathbf{E}_2(x + v_p t). \quad (5.12)$$

We now generalize the problem to cases where the wave vector is oriented along an arbitrary vector $\mathbf{k} = k \mathbf{n}$, and the associated plane wave satisfies both the wave equation and Maxwell's equations. We will continue to use the complex exponential

representation for the electric and magnetic induction fields, as in equation (5.7), with the understanding that a physical field is obtained by taking the real part of the exponential. With

$$\begin{aligned}\mathbf{E}(\mathbf{x}, t) &= \mathcal{E}e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)} \\ \mathbf{B}(\mathbf{x}, t) &= \mathcal{B}e^{i(k\mathbf{n}\cdot\mathbf{x}-\omega t)},\end{aligned}\tag{5.13}$$

where \mathcal{E} , \mathcal{B} , and \mathbf{n} are vectors independent of \mathbf{x} and t . Inserting these equations into their respective wave equations (i.e., equations (5.5) and (5.6)) yields

$$k^2\mathbf{n}\cdot\mathbf{n} = \omega^2\mu\epsilon,\tag{5.14}$$

with $\mathbf{n}\cdot\mathbf{n} = 1$, so that equation (5.8) is still applicable. Using Maxwell's equations, we find from the divergence-less conditions that

$$\boxed{\begin{aligned}\mathbf{n}\cdot\mathcal{E} &= 0 \\ \mathbf{n}\cdot\mathcal{B} &= 0\end{aligned}}\tag{5.15}$$

which implies that the electric and magnetic induction fields associated to a plane wave are perpendicular to the direction of propagation \mathbf{n} . Such a wave is called a **transverse wave**. The other two of Maxwell's equations give

$$\begin{aligned}\mathbf{n}\times\mathcal{E} &= \frac{\omega}{k}\mathcal{B} \\ \mathbf{n}\times\mathcal{B} &= -\mu\epsilon\frac{\omega}{k}\mathcal{E},\end{aligned}\tag{5.16}$$

which are really only one and the same. They bring the following additional condition on the relative orientation of the fields and \mathbf{n}

$$\boxed{\mathbf{n}\times\mathcal{E} = \frac{1}{\sqrt{\mu\epsilon}}\mathcal{B}}\tag{5.17}$$

It follows from this result that \mathbf{E} and \mathbf{B} are not only perpendicular to \mathbf{n} , but they are also perpendicular to each other. Additionally, in free space $1/\sqrt{\mu\epsilon} = c$ and, therefore, not only do $c\mathbf{B}$ and \mathbf{E} have the same units, but they also have the same magnitude

$$c|\mathbf{B}| = |\mathbf{E}|.\tag{5.18}$$

It is also common to define the **impedance** Z of a medium as

$$Z = \sqrt{\frac{\mu}{\epsilon}}, \quad (5.19)$$

then

$$\mathbf{n} \times \mathcal{E} = Z\mathcal{H}, \quad (5.20)$$

with $\mathbf{H} = \mathcal{H}e^{i(\mathbf{k}\mathbf{n}\cdot\mathbf{x}-\omega t)}$ for the magnetic field. The impedance of free space is $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377$ ohms. It is easy to show that when using complex fields (as in equations (5.13)) that the **time-averaged flux of energy**, which must be real, is given by

$$\begin{aligned} \mathbf{S} &= \frac{1}{2} \mathbf{E} \times \mathbf{H}^* \\ &= \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} |\mathcal{E}|^2 \mathbf{n}. \end{aligned} \quad (5.21)$$

The **time-averaged energy density** is

$$\begin{aligned} u &= \frac{1}{4} \left(\epsilon \mathbf{E} \cdot \mathbf{E}^* + \frac{1}{\mu} \mathbf{B} \cdot \mathbf{B}^* \right) \\ &= \frac{\epsilon}{2} |\mathcal{E}|^2. \end{aligned} \quad (5.22)$$

It is important to note that the condition $\mathbf{n} \cdot \mathbf{n} = 1$ leaves open the possibility that \mathbf{n} could be a complex vector. In that case, the exponential part of the wave becomes

$$e^{i(\mathbf{k}\mathbf{n}\cdot\mathbf{x}-\omega t)} = e^{-k\mathbf{n}_I \cdot \mathbf{x}} e^{i(k\mathbf{n}_R \cdot \mathbf{x}-\omega t)}, \quad (5.23)$$

with $\mathbf{n} = \mathbf{n}_R + i\mathbf{n}_I$. The wave is seen to either grow or decay in some directions. The real and imaginary vectors component of \mathbf{n} are linked through the relations

$$\mathbf{n} \cdot \mathbf{n} = n_R^2 - n_I^2 + 2i\mathbf{n}_R \cdot \mathbf{n}_I = 1, \quad (5.24)$$

or equivalently

$$\begin{aligned} n_R^2 - n_I^2 &= 1 \\ \mathbf{n}_R \cdot \mathbf{n}_I &= 0. \end{aligned} \quad (5.25)$$

A vector \mathbf{n} that satisfy equations (5.25) can be written as

$$\mathbf{n} = \cosh(\theta)\mathbf{e}_1 + i \sinh(\theta)\mathbf{e}_2, \quad (5.26)$$

where \mathbf{e}_1 and \mathbf{e}_2 are real unit basis vectors (so is \mathbf{e}_3 below), and θ is a real constant. Since the relation $\mathbf{n} \cdot \boldsymbol{\mathcal{E}} = 0$ must also be satisfied, the most general solution for the electric field is therefore

$$\boldsymbol{\mathcal{E}} = A [i \sinh(\theta) \mathbf{e}_1 - \cosh(\theta) \mathbf{e}_2] + A' \mathbf{e}_3, \quad (5.27)$$

with A and A' some complex constants. Therefore, we see that in general the electric field can have a component in the direction of propagation (i.e., \mathbf{n}_R), unlike for plane waves.

5.2 Polarization and the Stokes Parameters

We now go back to the propagation of plane waves, and define two waves propagating in the same direction \mathbf{n} with

$$\begin{aligned} \mathbf{E}_1 &= \mathbf{e}_1 E_1 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\ \mathbf{E}_2 &= \mathbf{e}_2 E_2 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)} \\ \mathbf{B}_j &= \sqrt{\mu \epsilon} \mathbf{n} \times \mathbf{E}_j, \end{aligned} \quad (5.28)$$

where $j = 1$ or 2 , the constants E_1 and E_2 are complex, and $\mathbf{k} = k\mathbf{n}$. The superposition of these electromagnetic fields can be interpreted as a single wave characterized by a total electric field $\mathbf{E}(\mathbf{x}, t)$

$$\mathbf{E}(\mathbf{x}, t) = (\mathbf{e}_1 E_1 + \mathbf{e}_2 E_2) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}. \quad (5.29)$$

It is important to note that the fact that E_1 and E_2 are complex can bring phase differences between the two components of the total electric field. In general, a phase difference will render the overall wave **elliptically polarized**. That is, the tip of the electric field vector is seen to be describing the figure of an ellipse over the interval of a period ($= 2\pi/\omega$). An important case is that of **linear polarization**, which happens when the two components share the same phase. It is then seen that the total electric field will be oriented at an angle

$$\theta = \tan^{-1} \left(\frac{\operatorname{Re}(E_2)}{\operatorname{Re}(E_1)} \right) \quad (5.30)$$

relative to the \mathbf{e}_1 unit basis vector. On the other hand, we have **circular polarization** when E_1 and E_2 have the same amplitude, but are out-of-phase by $\pm\pi/2$. Equation (5.29) is then written as

$$\mathbf{E}(\mathbf{x}, t) = E_1 (\mathbf{e}_1 \pm i \mathbf{e}_2) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}. \quad (5.31)$$

Taking the real part for both components, we find that the angle made by the electric field in relation to the \mathbf{e}_1 -axis is a function of time with

$$\begin{aligned}\theta(t) &= \tan^{-1} \left[\frac{\operatorname{Re}(E_2)}{\operatorname{Re}(E_1)} \right] \\ &= \mp \tan^{-1} \left[\frac{\sin(\mathbf{k} \cdot \mathbf{x} - \omega t)}{\cos(\mathbf{k} \cdot \mathbf{x} - \omega t)} \right] \\ &= \mp(\mathbf{k} \cdot \mathbf{x} - \omega t).\end{aligned}\tag{5.32}$$

At a given stationary point \mathbf{x} in space, the electric field will sweep a circle at the angular frequency ω . If, for example, we set $\mathbf{e}_1 = \mathbf{e}_x$, $\mathbf{e}_2 = \mathbf{e}_y$, and $\mathbf{n} = \mathbf{e}_z$, then at a point z the wave is said to be right circularly polarized for the “+” sign in equation (5.32) (“−” in equation (5.31)), and left circularly polarized for the “−” (“+” in equation (5.31)), as seen by an observer facing the incoming wave.

Just as the unit vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ form a basis for a plane wave, an alternative basis can be built using the *circularly polarized basis vectors* \mathbf{e}_+ and \mathbf{e}_- , defined with

$$\mathbf{e}_{\pm} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2),\tag{5.33}$$

and the vector \mathbf{n} . That these three vectors form a complete basis can be verified with the following relations

$$\begin{aligned}\mathbf{e}_{\pm}^* \cdot \mathbf{e}_{\mp} &= 0 \\ \mathbf{e}_{\pm}^* \cdot \mathbf{n} &= 0 \\ \mathbf{e}_{\pm}^* \cdot \mathbf{e}_{\pm} &= 1,\end{aligned}\tag{5.34}$$

and any vector, such as the electric vector, can be written using this basis

$$\mathbf{E}(\mathbf{x}, t) = (E_+ \mathbf{e}_+ + E_- \mathbf{e}_-) e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)},\tag{5.35}$$

with E_{\pm} the complex amplitudes.

The **Stokes parameters** are a set of four parameters that allow for the determination of the polarization characteristics of a beam of radiation, which is, of course, nothing more than a superposition of plane waves (through the Fourier transform). The amplitudes of radiation along the different basis vectors (perpendicular to the direction of propagation) are given by $\mathbf{e}_1 \cdot \mathbf{E}$ and $\mathbf{e}_2 \cdot \mathbf{E}$ for linear polarizations along \mathbf{e}_1 , and \mathbf{e}_2 , and $\mathbf{e}_+^* \cdot \mathbf{E}$ and $\mathbf{e}_-^* \cdot \mathbf{E}$ for left (\mathbf{e}_+) and right (\mathbf{e}_-) circular polarizations, respectively. Please

note the use of the complex conjugate, as is usually done for the scalar product between complex quantities. If we define these amplitudes, using equations (5.31) and (5.35), as

$$\begin{aligned} E_1 &= a_1 e^{i\delta_1}, & E_2 &= a_2 e^{i\delta_2} \\ E_+ &= a_+ e^{i\delta_+}, & E_- &= a_- e^{i\delta_-} \end{aligned} \quad (5.36)$$

then the Stokes parameters are given by

$$\begin{aligned} I &= |\mathbf{e}_1 \cdot \mathbf{E}|^2 + |\mathbf{e}_2 \cdot \mathbf{E}|^2 = a_1^2 + a_2^2 \\ Q &= |\mathbf{e}_1 \cdot \mathbf{E}|^2 - |\mathbf{e}_2 \cdot \mathbf{E}|^2 = a_1^2 - a_2^2 \\ U &= 2 \operatorname{Re} \left[(\mathbf{e}_1 \cdot \mathbf{E})^* (\mathbf{e}_2 \cdot \mathbf{E}) \right] = 2a_1 a_2 \cos(\delta_2 - \delta_1) \\ V &= 2 \operatorname{Im} \left[(\mathbf{e}_1 \cdot \mathbf{E})^* (\mathbf{e}_2 \cdot \mathbf{E}) \right] = 2a_1 a_2 \sin(\delta_2 - \delta_1) \end{aligned} \quad (5.37)$$

when using the $(\mathbf{e}_+, \mathbf{e}_-)$ basis, and by

$$\begin{aligned} I &= |\mathbf{e}_+^* \cdot \mathbf{E}|^2 + |\mathbf{e}_-^* \cdot \mathbf{E}|^2 = a_+^2 + a_-^2 \\ Q &= 2 \operatorname{Re} \left[(\mathbf{e}_+^* \cdot \mathbf{E})^* (\mathbf{e}_-^* \cdot \mathbf{E}) \right] = 2a_+ a_- \cos(\delta_- - \delta_+) \\ U &= 2 \operatorname{Im} \left[(\mathbf{e}_+^* \cdot \mathbf{E})^* (\mathbf{e}_-^* \cdot \mathbf{E}) \right] = 2a_+ a_- \sin(\delta_- - \delta_+) \\ V &= |\mathbf{e}_+^* \cdot \mathbf{E}|^2 - |\mathbf{e}_-^* \cdot \mathbf{E}|^2 = a_+^2 - a_-^2 \end{aligned} \quad (5.38)$$

when using $(\mathbf{e}_+, \mathbf{e}_-)$ basis. It is apparent from the second of equations (5.37) that Q is a measure of the amount linear polarization, and from the last of equations (5.38) that V is a measure of the amount of circular polarization. Both sets make it clear that I is the total intensity of radiation, irrespective of polarization. It can be shown (see the third list of problems) that U is also a measure of linear polarization, but along axes different from \mathbf{e}_1 and \mathbf{e}_2 . It can easily be verified from equations (5.37) and (5.38) that the intensity of radiation I is related to the other Stokes parameters by

$$I^2 = Q^2 + U^2 + V^2. \quad (5.39)$$

It is important to realize, however, that this last relation is only valid for purely deterministic and monochromatic signals. In general, radiation comes in the form of beams (as opposed to a single plane wave) and are, therefore, not monochromatic, and can also only be partially polarized (either linearly, circularly, or both). Then, equation (5.39) must in general be replaced with

$$I^2 \geq Q^2 + U^2 + V^2. \quad (5.40)$$

5.3 Frequency Dispersion in Dielectrics, Conductors, and Plasmas

5.3.1 Frequency Dependency of Dielectrics

Let us consider a medium of low density made of molecules for which we model any bound electron by a harmonic force. Under the action of an electric field the equations of motion for the electron is therefore

$$m(\ddot{\mathbf{x}} + \gamma\dot{\mathbf{x}} + \omega_0^2\mathbf{x}) = -q\mathbf{E}(\mathbf{x}, t), \quad (5.41)$$

where ω_0 and γ are, respectively, the natural frequency and the damping factor associated with the harmonic oscillator. Taking the Fourier transform of either sides of equation (5.41), we get (the Fourier transform of the n^{th} order time derivative of a function equals the Fourier transform of the function times $(-i\omega)^n$)

$$m\mathbf{x}(\omega_0^2 - \omega^2 - i\gamma\omega) = -q\mathbf{E}(\mathbf{x}, \omega). \quad (5.42)$$

The electric dipole associated with the bounded electron is

$$\mathbf{p} = -q\mathbf{x} = \frac{q^2}{m} \frac{\mathbf{E}(\mathbf{x}, \omega)}{(\omega_0^2 - \omega^2 - i\gamma\omega)}. \quad (5.43)$$

If a molecule has on average Z such electrons, out of which f_j have a binding frequency ω_j , then the electric dipole becomes

$$\mathbf{p} = \frac{q^2}{m} \sum_j \frac{f_j \mathbf{E}(\mathbf{x}, \omega)}{(\omega_j^2 - \omega^2 - i\gamma_j\omega)}, \quad (5.44)$$

with the following restriction on the **oscillator strengths** f_j

$$\sum_j f_j = Z. \quad (5.45)$$

Moreover, if the medium has are N molecules per unit volume, then its polarization is $\mathbf{P} = N\mathbf{p}$, and from equations (2.141) and (2.143) we can write

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}(\mathbf{x}, \omega), \quad (5.46)$$

and

$$\begin{aligned}\frac{\varepsilon(\omega)}{\varepsilon_0} &= 1 + \chi_e \\ &= 1 + \frac{Nq^2}{\varepsilon_0 m} \sum_j \frac{f_j}{(\omega_j^2 - \omega^2 - i\gamma_j \omega)},\end{aligned}\tag{5.47}$$

from equation (5.44).

5.3.2 Anomalous Dispersion and Resonant Absorption

Since the binding (or resonant) frequencies ω_j are usually much greater than the damping factors γ_j for typical molecules, we see from equation (5.47) for the normalized permittivity that each term in the summation will be real positive when $\omega < \omega_j$, and real negative when $\omega > \omega_j$. We deduce from this that $\varepsilon(\omega) > \varepsilon_0$ for low enough frequencies, while $\varepsilon(\omega) < \varepsilon_0$ when the frequency is high enough that many terms in the summation contribute negatively to the permittivity. On the other hand, for frequencies neighboring a resonant frequency ω_j , the corresponding term of the summation will dominate in equation (5.47), and the permittivity becomes complex and undergoes drastic changes in its value. One such feature is the change of sign of the derivative of the real component of the permittivity as the frequency crosses a resonance. We have a so-called **normal dispersion** when $d\text{Re}[\varepsilon(\omega)]/d\omega > 0$, and **anomalous dispersion** when $d\text{Re}[\varepsilon(\omega)]/d\omega < 0$. Normal dispersion happens everywhere on either side of a resonance while anomalous dispersion happens at a resonance.

Since the permittivity becomes complex at a resonance, there will be dissipation of electromagnetic energy from a wave as it travels through the medium; this phenomenon is called **resonant absorption**. If the intensity ($\propto E^2$) decreases as $e^{-\alpha z}$ as it propagates, then we can generally define the wave number with

$$k = \beta + i\frac{\alpha}{2}.\tag{5.48}$$

We already know from equation (5.9) that (in non-magnetic media $\mu = \mu_0$)

$$k = \omega\sqrt{\mu_0\varepsilon},\tag{5.49}$$

and therefore

$$\begin{aligned}k^2 &= \beta^2 - \frac{\alpha^2}{4} + i\alpha\beta \\ &= \omega^2\mu_0\varepsilon_0 \frac{\varepsilon(\omega)}{\varepsilon_0},\end{aligned}\tag{5.50}$$

or

$$\begin{aligned}\beta^2 - \frac{\alpha^2}{4} &= \frac{\omega^2}{c^2} \operatorname{Re} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} \right] \\ \alpha\beta &= \frac{\omega^2}{c^2} \operatorname{Im} \left[\frac{\varepsilon(\omega)}{\varepsilon_0} \right].\end{aligned}\tag{5.51}$$

5.3.3 The Low-frequency limit and the Electric Conductivity

Good conductors can be modeled as possessing free electrons, i.e., electrons that are not bound harmonically to the nuclei of the molecules, for which the resonant frequency is zero. If the fraction of free electrons is f_0 , then we can write the permittivity of equation (5.47) as

$$\varepsilon(\omega) = \varepsilon_b(\omega) + i \frac{Nq^2 f_0}{m\omega(\gamma_0 - i\omega)},\tag{5.52}$$

where $\varepsilon_b(\omega)$ is the permittivity due to all the other bound electrons, plus ε_0 .

Alternatively, we could model the conducting medium as obeying **Ohm's law**

$$\mathbf{J} = \sigma \mathbf{E},\tag{5.53}$$

while having a permittivity $\varepsilon_b(\omega)$, and use the Ampère-Maxwell equation to write

$$\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J} - i\omega \mathbf{D} \\ &= [\sigma - i\omega \varepsilon_b(\omega)] \mathbf{E} \\ &= -i\omega \left[\varepsilon_b(\omega) + i \frac{\sigma}{\omega} \right] \mathbf{E} \\ &= -i\omega \varepsilon(\omega) \mathbf{E}.\end{aligned}\tag{5.54}$$

Comparing equations (5.52) and (5.54) we evaluate the electrical conductivity of the medium to be

$$\sigma = \frac{f_0 N q^2}{m(\gamma_0 - i\omega)}.\tag{5.55}$$

As an example, copper has $N \approx 8 \times 10^{28}$ atoms/m³, with a low-frequency conductivity of $\sigma \approx 5.9 \times 10^7$ ($\Omega \cdot \text{m}$)⁻¹. This yields a damping factor of $\gamma_0/f_0 \approx 4 \times 10^{13}$ s⁻¹. If we assume that $f_0 \sim 1$, then the conductivity is seen to be real for frequencies of up to a few terahertz. Only beyond these frequencies will it become complex.

5.3.4 The High-frequency limit and the Plasma Frequency

The high-frequency limit is interpreted as the domain of frequencies located well beyond the highest resonant frequency in equation (5.47). For such frequencies, this equation, which expresses the permittivity as a function of frequency, can be approximated to

$$\begin{aligned}\frac{\varepsilon(\omega)}{\varepsilon_0} &\approx 1 - \frac{Nq^2}{\varepsilon_0 m} \sum_j \frac{f_j}{\omega^2} \\ &\approx 1 - \frac{\omega_p^2}{\omega^2},\end{aligned}\tag{5.56}$$

with the so-called **plasma frequency** ω_p is defined by

$$\omega_p^2 = \frac{NZq^2}{\varepsilon_0 m}.\tag{5.57}$$

We can use equation (5.49) to evaluate the dispersion relation, i.e., the wave number as a function of the frequency,

$$\begin{aligned}k^2 &= \omega^2 \mu_0 \varepsilon(\omega) \\ &= \omega^2 \mu_0 \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right) \\ &= \frac{\omega^2 - \omega_p^2}{c^2},\end{aligned}\tag{5.58}$$

or,

$$k = \frac{1}{c} \sqrt{\omega^2 - \omega_p^2}.\tag{5.59}$$

We note for this last result that for frequencies lower than the plasma frequency (i.e., $\omega < \omega_p$) the wave number is completely imaginary, and a plane wave will not be able to propagate through the medium.

5.4 The Group Velocity

We already know that for a monochromatic wave (such as a wave plane, for example) the **phase velocity**, defined by

$$v_p = \frac{\omega}{k},\tag{5.60}$$

is the velocity of propagation. If, however, we superpose many waves of different frequencies, forming what is called a **wave packet**, then the velocity of propagation is not given by the phase velocity, as this quantity may be different for each wave if the medium is dispersive, but by the so-called **group velocity**. To see how this comes about, let us suppose that the medium is indeed dispersive with some well-defined relation

$$\omega = \omega(k), \quad (5.61)$$

which we expand around some value k_0 . Then

$$\omega(k) = \omega_0 + \left. \frac{d\omega}{dk} \right|_{k_0} (k - k_0) + \dots \quad (5.62)$$

Since our wave, let us call it $u(x,t)$ (we work in one-dimension for simplicity), is composed of a superposition of plane waves, we can expand with a Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega t)} dk \quad (5.63)$$

with

$$A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx. \quad (5.64)$$

Inserting equation (5.62) into equation (5.63), we find

$$u(x,t) \simeq \frac{e^{i[k_0(d\omega/dk)|_{k_0} - \omega_0]t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i[x - t(d\omega/dk)|_{k_0}]k} dk. \quad (5.65)$$

But also from equation (5.63), it is apparent that

$$u(x,t) \simeq u\left(x - t \left. \frac{d\omega}{dk} \right|_{k_0}, 0\right) e^{i[k_0(d\omega/dk)|_{k_0} - \omega_0]t}. \quad (5.66)$$

It follows from equation (5.66) that, apart from the phase factor, the wave packet travels, with its shape seemingly unaltered, at the **group velocity**

$$v_g = \left. \frac{d\omega}{dk} \right|_{k_0}. \quad (5.67)$$

This implies that if energy is associated with the amplitude of the wave, then energy transport will also happen at the group velocity.

The preceding treatment was, however, limited in its generality by the assumption that the dispersion relation is approximately linear. We could improve the precision in the Taylor expansion of equation (5.62) to show that the wave packet will actually spread as it travels at the group velocity. This fact can be more simply ascertained using the wave train uncertainty relation. That is, if at some time t_0 the wave packet has a width Δx_0 , then its “spectral” extent in k -space is given by

$$\Delta k \sim \frac{1}{\Delta x_0}. \quad (5.68)$$

The corresponding spread, or variation in the group velocity is given by

$$\Delta v_g \sim \left. \frac{d^2 \omega}{dk^2} \right|_{k_0} \Delta k \sim \frac{1}{\Delta x_0} \left. \frac{d^2 \omega}{dk^2} \right|_{k_0}. \quad (5.69)$$

The uncertainty in position at a later time t will be obtained by adding the square of the initial uncertainty with the square of the uncertainty that arises from equation (5.69) as time progresses and the wave travels. That is,

$$\begin{aligned} \Delta x(t) &= \sqrt{(\Delta x_0)^2 + (\Delta v_g t)^2} \\ &= \sqrt{(\Delta x_0)^2 + \left(\frac{t}{\Delta x_0} \left. \frac{d^2 \omega}{dk^2} \right|_{k_0} \right)^2}. \end{aligned} \quad (5.70)$$

This result makes it clear that we should expect the wave packet to spread as it propagates at the group velocity. This effect is increasingly important as the initial width of the wave packet decreases.

(Incidentally, equation (5.68) can easily be verified using the Fourier transform pair. If we consider, for example, a pulse given by

$$g(x) = \begin{cases} B, & \text{for } |x| < \frac{\epsilon}{2} \\ 0, & \text{for } |x| > \frac{\epsilon}{2} \end{cases} \quad (5.71)$$

Then its Fourier transform is

$$\begin{aligned}
G(k) &= \frac{B}{\sqrt{2\pi}} \int_{-\varepsilon/2}^{\varepsilon/2} e^{-ikx} dx \\
&= \frac{1}{\sqrt{2\pi}} \frac{B}{ik} \left(e^{ik\frac{\varepsilon}{2}} - e^{-ik\frac{\varepsilon}{2}} \right) \\
&= \frac{2B}{\sqrt{2\pi}} \frac{\sin\left(k\frac{\varepsilon}{2}\right)}{k}.
\end{aligned} \tag{5.72}$$

Since the first zeros of $G(k)$ happen at $|k|=2\pi/\varepsilon$, we see that equation (5.68) is approximately recovered. Moreover, this relation (i.e., equation (5.68)) could have been easily guessed from the Heisenberg uncertainty

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}, \tag{5.73}$$

since this relation can be transformed to

$$\Delta x \cdot \Delta k \geq \frac{1}{2}, \tag{5.74}$$

with a simple division by Planck's constant.)

5.5 The High-frequency Limit Revisited – Electromagnetic Oscillations in Cold Plasmas

We now go back to the high-frequency limit discussed in section 5.3.4, while modifying our treatment to apply it more directly to the case of a plasma (such as the ionosphere, for example). A plasma is, in general, a gas for which the constituent particles are charged. We will in what follows, however, limit ourselves to the study of so-called cold plasmas for which there are no pressure gradients that can also act as forces (or, more precisely, force densities) on the gas.

In particular, we consider perturbations in an initially uniform and homogeneous plasma. Although electron and positive ion densities are both components of the plasma, we will neglect the presence of the latter since ions are much heavier than electrons, and move much more slowly than the electrons when subjected to high-frequency electromagnetic fields. So, let the number density of the electrons be $n_0 + n_1$ (with n_0 the initial density, n_1 the perturbation, and $n_0 \gg n_1$), and the perturbations on their velocity, electric field, and magnetic induction field be $\mathbf{v}_1, \mathbf{E}_1, \mathbf{B}_1$, respectively. It follows from our treatment that these last three vector quantities are small (this is what is meant by “perturbations”), and that their corresponding initial values $\mathbf{v}_0, \mathbf{E}_0, \mathbf{B}_0$ are all zero. With m the mass of the electron, we can write the equation of motion as

$$mn_0 \frac{d\mathbf{v}_1}{dt} = -qn_0 (\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_1). \quad (5.75)$$

We know, however, from equation (3.91) that the Lagrangian derivative present in equation (5.75) should be written as

$$\frac{d\mathbf{v}_1}{dt} = \frac{\partial \mathbf{v}_1}{\partial t} + (\mathbf{v}_1 \cdot \nabla) \mathbf{v}_1, \quad (5.76)$$

but since we are considering perturbations, we will linearize (i.e., keeping only lowest order terms) all our equations and write

$$\frac{d\mathbf{v}_1}{dt} \approx \frac{\partial \mathbf{v}_1}{\partial t}, \quad (5.77)$$

and also

$$\mathbf{E}_1 \approx \mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_1. \quad (5.78)$$

The equation of motion simplifies to

$$mn_0 \frac{\partial \mathbf{v}_1}{\partial t} = -qn_0 \mathbf{E}_1. \quad (5.79)$$

We combine to equation (5.79) the Faraday law of induction

$$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_1}{\partial t}, \quad (5.80)$$

and the Ampère-Maxwell law

$$\begin{aligned} \nabla \times \mathbf{B}_1 &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}_1}{\partial t} \\ &= -\mu_0 qn_0 \mathbf{v}_1 + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}_1}{\partial t} \\ &= \frac{1}{c^2} \left(\frac{\partial \mathbf{E}_1}{\partial t} - \frac{qn_0}{\epsilon_0} \mathbf{v}_1 \right). \end{aligned} \quad (5.81)$$

As before we are looking for solution involving plane waves with a dependence of the form $e^{-i\omega t}$, and we therefore replace $\partial/\partial t$ by $-i\omega$. Then equation (5.79) becomes

$$\mathbf{v}_1 = \frac{q}{i\omega m} \mathbf{E}_1, \quad (5.82)$$

and equation (5.81) can be rewritten (after inserting equation (5.82) into it)

$$\nabla \times \mathbf{B}_1 = -i\omega\mu_0\varepsilon\mathbf{E}_1, \quad (5.83)$$

with the frequency dependent permittivity of the plasma

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2} \right). \quad (5.84)$$

Again, ω_p is the plasma frequency, and is equivalent to our previous definition (see equation (5.57)) with

$$\omega_p^2 = \frac{q^2 n_0}{m\varepsilon_0}. \quad (5.85)$$

We now take the curl of equation (5.80), to get

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}_1) &= i\omega(\nabla \times \mathbf{B}_1) \\ &= \omega^2\mu_0\varepsilon\mathbf{E}_1, \end{aligned} \quad (5.86)$$

and if we further assume that every perturbation field will be of the form $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$, then

$$\mathbf{k} \times (\mathbf{k} \times \mathbf{E}_1) = -\omega^2\mu_0\varepsilon\mathbf{E}_1, \quad (5.87)$$

or,

$$\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) - \left[k^2 - \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right) \right] \mathbf{E}_1 = 0. \quad (5.88)$$

5.5.1 Plasma Oscillations

Equation (5.88) shows that in a direction parallel to the orientation of the wave vector an electric field component can exist, as long as

$$(\omega^2 - \omega_p^2)\mathbf{E}_1^{\parallel} = 0. \quad (5.89)$$

This implies that $\omega = \omega_p$, and the frequency of oscillation is that of the plasma frequency. However, since

$$v_g = \frac{d\omega}{dk} = 0, \quad (5.90)$$

then these oscillations represent non-propagating longitudinal **plasma** or **Langmuir oscillations**. That is, we are *not* in the presence of traveling electromagnetic waves.

5.5.2 Electromagnetic Waves

The only other possible solution to equation (5.88) is for the case when the electric field is perpendicular to the wave vector. Equation (5.88) becomes

$$\left[k^2 - \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right) \right] \mathbf{E}_1^\perp = 0, \quad (5.91)$$

with the following dispersion relation

$$\omega^2 = k^2 c^2 + \omega_p^2. \quad (5.92)$$

We find, therefore, find that the phase velocity $v_p = \omega/k$ is greater than the speed of light. This peculiar result is, however, not a problem since the wave will propagate at the group velocity

$$\begin{aligned} v_g &= \frac{d\omega}{dk} = \frac{kc^2}{\sqrt{k^2 c^2 + \omega_p^2}} \\ &= c \sqrt{1 - \frac{\omega_p^2}{\omega^2}}, \end{aligned} \quad (5.93)$$

which is lower than the speed of light. We see from equation (5.92) that the wave number becomes imaginary when the wave has a frequency such that $\omega < \omega_p$. Since this wave cannot travel in the plasma, then it can only be reflected when incident on its boundary.

5.6 Wave Propagation Parallel to a Magnetic Induction Field

We now complicate the problem by allowing for the presence of a uniform and constant magnetic induction field \mathbf{B}_0 . Under these conditions, the linearized equation of motion becomes

$$m n_0 \frac{\partial \mathbf{v}_1}{\partial t} = -q n_0 (\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0), \quad (5.94)$$

or equivalently

$$i\omega m \mathbf{v}_1 = q (\mathbf{E}_1 + \mathbf{v}_1 \times \mathbf{B}_0). \quad (5.95)$$

On the other hand, the equations for the Faraday and Ampère-Maxwell laws remain unchanged with

$$\begin{aligned}\nabla \times \mathbf{E}_1 &= i\omega \mathbf{B}_1 \\ \nabla \times \mathbf{B}_1 &= -\frac{i\omega}{c^2} \mathbf{E}_1 - \mu_0 q n_0 \mathbf{v}_1.\end{aligned}\tag{5.96}$$

If we break up the different vector quantities into two components, one parallel, and another perpendicular to \mathbf{B}_0 , then we find from equation (5.95)

$$\begin{aligned}\mathbf{v}_1^\parallel &= \frac{q}{i\omega m} \mathbf{E}_1^\parallel \\ \mathbf{v}_1^\perp &= \frac{q}{i\omega m} (\mathbf{E}_1^\perp + \mathbf{v}_1^\perp \times \mathbf{B}_0).\end{aligned}\tag{5.97}$$

But the last equation can be transformed by evaluating

$$\mathbf{v}_1^\perp \times \mathbf{B}_0 = \frac{q}{i\omega m} (\mathbf{E}_1^\perp \times \mathbf{B}_0 - \mathbf{v}_1^\perp B_0^2),\tag{5.98}$$

and upon inserting this result back in the last of equations (5.97)

$$\mathbf{v}_1^\perp = -i\omega \frac{q}{m} \frac{\left(\mathbf{E}_1^\perp + \mathbf{E}_1^\perp \times \frac{\boldsymbol{\omega}_c}{i\omega} \right)}{(\omega^2 - \omega_c^2)},\tag{5.99}$$

with the **cyclotron frequency** vector

$$\boldsymbol{\omega}_c = \frac{q\mathbf{B}_0}{m}.\tag{5.100}$$

Inserting these results in the second of equations (5.96), we have

$$\begin{aligned}(\nabla \times \mathbf{B}_1)_\parallel &= -\frac{i\omega}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right) \mathbf{E}_1^\parallel \\ (\nabla \times \mathbf{B}_1)_\perp &= -\frac{i\omega}{c^2} \left[\left(1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \right) \mathbf{E}_1^\perp - \frac{i\omega_p^2}{\omega^2 - \omega_c^2} \left(\frac{\boldsymbol{\omega}_c}{\omega} \times \mathbf{E}_1^\perp \right) \right].\end{aligned}\tag{5.101}$$

The notation for these seemingly complicated equations can be simplified if we introduce a second-rank tensor $\bar{\boldsymbol{\epsilon}}$ for the permittivity of the plasma, then

$$\nabla \times \mathbf{B}_1 = -i\omega\mu_0(\tilde{\boldsymbol{\epsilon}} \cdot \mathbf{E}_1). \quad (5.102)$$

For example, if $\mathbf{B}_0 = B_0 \mathbf{e}_z$, then

$$\tilde{\boldsymbol{\epsilon}} = \epsilon_0 \begin{bmatrix} h_1 & ih_2 & 0 \\ -ih_2 & h_1 & 0 \\ 0 & 0 & h_3 \end{bmatrix}, \quad (5.103)$$

with

$$\begin{aligned} h_1 &= 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \\ h_2 &= \frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_c^2} \\ h_3 &= 1 - \frac{\omega_p^2}{\omega^2}. \end{aligned} \quad (5.104)$$

We now take the curl of the Faraday law (i.e. the first of equations (5.96)), and substitute equation (5.102) to obtain

$$\nabla \times (\nabla \times \mathbf{E}_1) = \omega^2 \mu_0 (\tilde{\boldsymbol{\epsilon}} \cdot \mathbf{E}_1), \quad (5.105)$$

or

$$k^2 \mathbf{E}_1 - \left[\mathbf{k}(\mathbf{k} \cdot \mathbf{E}_1) + \frac{\omega^2}{c^2} \left(\frac{\tilde{\boldsymbol{\epsilon}}}{\epsilon_0} \cdot \mathbf{E}_1 \right) \right] = 0. \quad (5.106)$$

The general solution of equation (5.106) for electromagnetic waves propagating in an arbitrary direction relative to the magnetic field is very complicated, so we will restrict ourselves to cases of wave propagation (specified by \mathbf{k}) parallel to the orientation of \mathbf{B}_0 . If we set, one more time, $\mathbf{B}_0 = B_0 \mathbf{e}_z$ (also $\mathbf{k} = k \mathbf{e}_z$, and $\boldsymbol{\omega}_c = \omega_c \mathbf{e}_z$), then equation (5.106) can be written in a matrix form as

$$\begin{bmatrix} 1 - \frac{\omega^2 h_1}{k^2 c^2} & -\frac{\omega^2 i h_2}{k^2 c^2} & 0 \\ \frac{\omega^2 i h_2}{k^2 c^2} & 1 - \frac{\omega^2 h_1}{k^2 c^2} & 0 \\ 0 & 0 & -\frac{\omega^2 h_3}{k^2 c^2} \end{bmatrix} \begin{pmatrix} E_1^x \\ E_1^y \\ E_1^z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (5.107)$$

A non-trivial solution to equation (5.107) will be obtained when the determinant of the matrix is set to zero. One root of the determinant gives the simple solution of a single non-zero electric field component along the direction of propagation ($E_1^z \neq 0$, and $E_1^x = E_1^y = 0$) with $h_3 = 0$. This is the plasma oscillation condition we found in section 5.5.1. More interestingly, the two other roots of the determinant can be evaluated with

$$\left(1 - \frac{\omega^2 h_1}{k^2 c^2}\right)^2 - \frac{\omega^4 h_2^2}{k^4 c^4} = 0, \quad (5.108)$$

or

$$\frac{k^2 c^2}{\omega^2} = h_1 \mp h_2. \quad (5.109)$$

The corresponding electric field vectors (i.e., the eigenvectors) associated with these dispersion relations are orthogonal to the direction of propagation, and are given by

$$\mathbf{E}_{\pm} = E_1 (\mathbf{e}_x \pm i \mathbf{e}_y) e^{i(k_{\pm} z - \omega t)}, \quad (5.110)$$

with E_1 a constant, and

$$k_{\pm} = \frac{\omega}{c} \sqrt{1 - \frac{\omega_p^2}{\omega(\omega \mp \omega_c)}}. \quad (5.111)$$

Take note that the “+” signs in equations (5.110) and (5.111) corresponds to the “−” sign of equation (5.109), and vice-versa. It is clear from equation (5.110) that the electric fields are circularly polarized (with the convention defined in section 5.2), and from equation (5.111) that the associated waves propagating parallel to the uniform magnetic induction field in the plasma will do so at different group velocities. We, therefore, conclude that the presence of the magnetic induction will not only render the plasma anisotropic (e.g., it has a different permittivity in directions parallel and perpendicular to \mathbf{B}_0), but also birefringent.

5.7 Magnetohydrodynamics

So far, we have neglected the presence of the more massive (and positively charged) ions when analyzing the behavior of an electron fluid in a plasma. In reality, one should set up a system of equations that would account for their presence and solve for both types of particles. To do so, we start with the following equations of motions

$$\begin{aligned}
m_e n \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e \right] &= -\nabla p_e - nq(\mathbf{E} + \mathbf{v}_e \times \mathbf{B}) - \mu n v_c (\mathbf{v}_e - \mathbf{v}_i) \\
m_i n \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla) \mathbf{v}_i \right] &= -\nabla p_i + nq(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}) - \mu n v_c (\mathbf{v}_i - \mathbf{v}_e),
\end{aligned} \tag{5.112}$$

for the electron and ion fluids, respectively (as implied by the subscripts for the different quantities). In equations (5.112) we have allowed for the presence of forces due pressure gradients and the friction brought by the collisions between the electrons and ions. The quantities q , μ , v_c , and n are, respectively, the elementary charge unit, the reduced mass ($\mu^{-1} = m_e^{-1} + m_i^{-1}$), the collision rate, and the number density (which is assumed to be the same for both types of charges). We could, in principle, solve a given problem for the electronic and ionic behavior with these equations (other equations are also needed; see below). But it is often simpler to combine the two fluids into a single fluid and treat the charged particles with only one equation of motion. [**Note:** This will be justified in cases when we consider scales where there is no charge separation (i.e., the fluid is neutral beyond a given length scale, the so-called Debye length) and changes happen over times that are much longer than the inverse of the plasma frequencies.]

We, therefore, define the mass density and velocity for our “single fluid” as

$$\begin{aligned}
\rho &\equiv n(m_e + m_i) \\
\mathbf{v} &\equiv \frac{m_e \mathbf{v}_e + m_i \mathbf{v}_i}{m_e + m_i},
\end{aligned} \tag{5.113}$$

and simply add the two equations of (5.112) and divide by $(m_e + m_i)$. When this done it is possible to show (see the third list of problems) that we are left with the following unique equation of motion

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \mathbf{J} \times \mathbf{B}, \tag{5.114}$$

where the current density \mathbf{J} was introduced with

$$\mathbf{J} = nq(\mathbf{v}_i - \mathbf{v}_e). \tag{5.115}$$

It is important to realize (again see the third list of problems) that the pressure appearing in equation (5.114) is *not* simply the sum of the electronic and ionic pressures. That is,

$$p \neq p_e + p_i. \tag{5.116}$$

Equation (5.114) is usually transformed using the modified Ampère’s Law from Maxwell’s equations with

$$\mathbf{J} = \frac{1}{\mu} \nabla \times \mathbf{B} - \varepsilon \frac{\partial \mathbf{E}}{\partial t}, \quad (5.117)$$

where it is assumed that the permittivity ε and the permeability μ are both constant and scalar quantities. But it probably comes with no surprise that the displacement current can be neglected in the analysis, since we are considering time variations that are long relative to the inverse of the plasma frequency. Alternatively, when we compare the two terms on the right hand side of equation (5.117) we have

$$\frac{\varepsilon |\partial \mathbf{E} / \partial t|}{\mu^{-1} |\nabla \times \mathbf{B}|} \sim \frac{\mu \varepsilon E/t}{B/l} \sim \frac{v}{c} \left(\frac{E}{cB} \right). \quad (5.118)$$

However, a somewhat generalized form of Ohm's Law that accounts for the presence of the magnetic induction field (compare with equation (5.53)) specifies that

$$\mathbf{J} = \sigma (\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (5.119)$$

and since the conductivity σ is expected to be very high (i.e., $\sigma \rightarrow \infty$) for a fluid made up of charged particles, then the current density can only remain finite if

$$\mathbf{E} \sim -\mathbf{v} \times \mathbf{B}. \quad (5.120)$$

Inserting this last relation into equation (5.118) we find that

$$\frac{\varepsilon |\partial \mathbf{E} / \partial t|}{\mu^{-1} |\nabla \times \mathbf{B}|} \sim \frac{v^2}{c^2}, \quad (5.121)$$

which further justifies our aforementioned approximation when restricting our analysis to non-relativistic motions. We therefore left with Ampère's Law

$$\mathbf{J} = \frac{1}{\mu} \nabla \times \mathbf{B}, \quad (5.122)$$

and a correspondingly transformed equation of motion

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p - \frac{\mathbf{B}}{\mu} \times (\nabla \times \mathbf{B}). \quad (5.123)$$

We can also add to equation (5.123) the relation for the conservation of mass (the continuity equation for the mass density and flux)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (5.124)$$

However, it is clear that these last two equations cannot form a complete set of equations that will allow us to solve our problem in general. This is because we are in the presence of eight unknown quantities (i.e., ρ , p , \mathbf{v} , and \mathbf{B}) while we only have four relations in equations (5.123) and (5.124) combined. We could also add to this an equation of state that will link, say, ρ (or the density n) and p , but we are still in need of three more equations to close our system. We can accomplish this as follows. First, we combine Ampère's Law (i.e., equation (5.122)) with Ohm's Law (i.e., equation (5.119)) to get

$$\mathbf{E} = \frac{1}{\mu\sigma} \nabla \times \mathbf{B} - \mathbf{v} \times \mathbf{B}, \quad (5.125)$$

and then we use Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (5.126)$$

to get

$$\begin{aligned} \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu\sigma} \nabla \times (\nabla \times \mathbf{B}) \\ &= \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu\sigma} [\nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}] \\ &= \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{\mu\sigma} \nabla^2 \mathbf{B}, \end{aligned} \quad (5.127)$$

where the quantity $1/(\mu\sigma)$ is often called the **magnetic diffusivity**. This last relation, the so-called **induction equation**, provides us with the three needed equations. We now have our full set of equations for magnetohydrodynamics (*remember that an equation of state is also needed*), which we now group together for convenience

$$\boxed{\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] + \nabla p + \frac{\mathbf{B}}{\mu} \times (\nabla \times \mathbf{B}) &= 0 \\ \frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{\mu\sigma} \nabla^2 \mathbf{B} &= 0 \end{aligned}} \quad (5.128)$$

And here lies the beauty and usefulness of the magnetohydrodynamics approximation. It provides an elegant and clean dynamical theory where everything is contained in a simple set of equations. One must always remember, however, the limitations of the

approximation. Mainly, it cannot be applied to high frequency phenomena (e.g., electromagnetic waves propagation), or cases where charge separation may arise.

One last comment before ending this section, it is interesting to note that we can transform the last term of the second of equations (5.128) with

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \nabla \left(\frac{B^2}{2} \right) - (\mathbf{B} \cdot \nabla) \mathbf{B}, \quad (5.129)$$

which when reinserted in the same equation yields

$$\rho \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] + \nabla \left(p + \frac{B^2}{2\mu} \right) - \frac{1}{\mu} (\mathbf{B} \cdot \nabla) \mathbf{B} = 0. \quad (5.130)$$

It is clear that the first term on the right hand side of equation (5.129) is to be interpreted as the **magnetic pressure**, while the other term can be shown to be of the nature of a tension along the magnetic field lines.

5.7.1 Hydromagnetic Waves

Just as we did in section 5.5, we now consider perturbations on an initially uniform and homogeneous plasma fluid, and we write the density, pressure, velocity, and magnetic field as $\rho_0 + \rho_1(\mathbf{x}, t)$, $p_0 + p_1(\mathbf{x}, t)$, $\mathbf{v}_1(\mathbf{x}, t)$, and $\mathbf{B}_0 + \mathbf{B}_1(\mathbf{x}, t)$, respectively. Also, we will assume that we are dealing with an adiabatic fluid (i.e., the perturbations in density and pressure evolve on times short compared to the typical time required for heat conduction, and we can approximate $dQ = 0$, where Q is the heat) with

$$p = K \rho^\gamma, \quad (5.131)$$

where K is a constant and γ is the ratio of the specific heats. We could simply modify this relation as follows

$$p = K \rho_0^\gamma \left(1 + \frac{\rho_1}{\rho_0} \right)^\gamma \approx K \rho_0^\gamma \left(1 + \gamma \frac{\rho_1}{\rho_0} \right), \quad (5.132)$$

or

$$p_1 = K \gamma \rho_0^{\gamma-1} \rho_1 = c_s^2 \rho_1, \quad (5.133)$$

with

$$c_s^2 = \frac{\gamma p_0}{\rho_0}, \quad (5.134)$$

and then insert it in the equation of motion (i.e., the second of equations (5.128)). But it is to our benefit to consider the behavior of such a fluid without the presence of a magnetic induction to get a better understanding of the corresponding terms. So, keeping things to the first order in the perturbations, we transform the continuity equation to

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0, \quad (5.135)$$

while the equation of motion (with $\mathbf{B} = 0$) becomes

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla p_1 = 0. \quad (5.136)$$

Inserting equation (5.133) in equation (5.136) we find

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + c_s^2 \nabla \rho_1 = 0, \quad (5.137)$$

which we put into equation (5.135) (after having taken its time derivative) to obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} - c_s^2 \nabla^2 \rho_1 = 0. \quad (5.138)$$

Equation (5.138) is a wave equation that describes the propagation of (longitudinal) acoustic perturbations in the fluid. Obviously, the parameter c_s is nothing else but the **sound speed** for the medium, which is generally expressed by

$$c_s^2 = \frac{dp}{d\rho}. \quad (5.139)$$

Proceeding further, we will also assume that the plasma has a very high conductivity and we neglect the last term on the left hand side of the induction equation of (5.128). We are now in a position to write down the set of linearized magnetohydrodynamics equations (from equations (5.128)) that characterize the behavior of the plasma under consideration

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 &= 0 \\ \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + c_s^2 \nabla \rho_1 + \frac{\mathbf{B}_0}{\mu} \times (\nabla \times \mathbf{B}_1) &= 0 \\ \frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) &= 0. \end{aligned} \quad (5.140)$$

On taking the time derivative of the second of equations (5.140), and inserting in it the other two equations of the same set, we find

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} - c_s^2 \nabla (\nabla \cdot \mathbf{v}_1) + \mathbf{v}_A \times \left\{ \nabla \times \left[\nabla \times (\mathbf{v}_1 \times \mathbf{v}_A) \right] \right\} = 0, \quad (5.141)$$

where we have introduced the so-called **Alfvén velocity**

$$\mathbf{v}_A = \frac{\mathbf{B}}{\sqrt{\mu \rho_0}}. \quad (5.142)$$

Although it not necessarily be obvious at this point, equation (5.141) is also a wave equation. We can verify this by considering an arbitrary Fourier component $\mathbf{v}_1 e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$ for the velocity field. Inserting this function in equation (5.142) yields

$$\begin{aligned} \omega^2 \mathbf{v}_1 &= c_s^2 \mathbf{k} (\mathbf{k} \cdot \mathbf{v}_1) - \mathbf{v}_A \times \left\{ \mathbf{k} \times \left[\mathbf{k} \times (\mathbf{v}_1 \times \mathbf{v}_A) \right] \right\} \\ &= c_s^2 \mathbf{k} (\mathbf{k} \cdot \mathbf{v}_1) - \mathbf{v}_A \times \left\{ \mathbf{k} \times \left[(\mathbf{k} \cdot \mathbf{v}_A) \mathbf{v}_1 - (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{v}_A \right] \right\} \\ &= c_s^2 \mathbf{k} (\mathbf{k} \cdot \mathbf{v}_1) - \mathbf{k} \left[(\mathbf{k} \cdot \mathbf{v}_A) (\mathbf{v}_1 \cdot \mathbf{v}_A) - (\mathbf{k} \cdot \mathbf{v}_1) v_A^2 \right] \\ &\quad + (\mathbf{k} \cdot \mathbf{v}_A) \left[(\mathbf{k} \cdot \mathbf{v}_A) \mathbf{v}_1 - (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{v}_A \right], \end{aligned} \quad (5.143)$$

and finally

$$\omega^2 \mathbf{v}_1 = (c_s^2 + v_A^2) \mathbf{k} (\mathbf{k} \cdot \mathbf{v}_1) + (\mathbf{k} \cdot \mathbf{v}_A) \left[(\mathbf{k} \cdot \mathbf{v}_A) \mathbf{v}_1 - (\mathbf{k} \cdot \mathbf{v}_1) \mathbf{v}_A - (\mathbf{v}_1 \cdot \mathbf{v}_A) \mathbf{k} \right]. \quad (5.144)$$

This last equation is the basic dispersion relation for hydromagnetic waves relating \mathbf{k} and ω when c_s and v_A are known. Although the solution of this equation is not straightforward, it allows for three different modes of propagation (see the third list of problems) of which one is easily determined. This simple solution is that of transverse waves perpendicular to the plane defined by the propagation vector \mathbf{k} and the Alfvén velocity \mathbf{v}_A . That is, setting $\mathbf{k} \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot \mathbf{v}_A = 0$ we find

$$\omega^2 = (\mathbf{k} \cdot \mathbf{v}_A)^2, \quad (5.145)$$

so that

$$\omega = \pm v_A k \cos(\theta), \quad (5.146)$$

where θ is the angle between the magnetic field and the propagation vector. The waves defined by equation (5.146) are called **Alfvén waves**, and suggest a picture of a magnetized plasma akin to that of a string on which transverse waves propagate. Following this line of thought, we see that any transverse perturbation on the magnetic

field lines would be opposed by the magnetic tension. Finally, it is noted from equation (5.146) that the group velocity of the wave is $v_A \cos(\theta)$.